

Is our Universe brany ?

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In brane-worlds, our universe is assumed to be a submanifold, or brane, embedded in a higher-dimensional bulk spacetime. Focusing on scenarios with a curved five-dimensional bulk spacetime, I discuss their gravitational and cosmological properties.

§1. Introduction

Brane-world models, which have received a lot of attention during the last few years, are essentially characterized by two basic ideas:

- they assume the existence of spatial extra dimensions, in addition to our four space-time dimensions. The higher dimensional spacetime is usually called the “bulk” spacetime;
- our accessible Universe is assumed to be a submanifold, called “brane”, embedded in the bulk spacetime. Ordinary matter fields are assumed to be confined on the brane.

This confinement of matter on a submanifold is really the novel ingredient that distinguishes brane-worlds from the more ancient models with extra-dimensions, based on the ideas of Kaluza and Klein.

The motivations for studying brane-worlds are multiple. First, branes appear in string/M theory. One example is the D-branes, corresponding to solitonic objects where open strings end. Another example is the two end-of-the-world branes of the Horawa-Witten model. Another interest is that some of the brane-world models have a strong link with the AdS/CFT correspondence. Brane-worlds have also turned to be very fruitful to address various questions of particle physics, in particular the hierarchy problem. Finally, brane-worlds are also particularly interesting for their gravitational properties.

In part because of this multiplicity of motivations, there exist many models of brane-worlds. It is often convenient to regroup them in two broad categories:

- models with compact flat extra dimensions
- models with warped extra-dimensions.

Another possible classification would be to distinguish models with so-called TeV gravity, i.e. for which the fundamental Planck mass is of the order of the TeV, from other models with a higher fundamental Planck mass.

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In this contribution, I will focus my attention on brane-worlds characterized by a single extra dimension where the bulk space-time is *curved* instead of flat and where the self-gravity of the brane is taken into account. This includes the configurations discussed by Randall and Sundrum. The present contribution is limited to a few selected topics. Other aspects and more details can be found in several introductory reviews.¹⁾

§2. Gravity in the brane

The main constraint on brane-worlds models is to recover usual gravity, at least approximately, in our brane-universe. Whereas the usual trick is to compactify the extra dimensions on a sufficiently small scale, another possibility was emphasized by Randall and Sundrum,²⁾ who considered *curved*, or *warped*, bulk geometries. They realized that *compact* extra-dimensions are not necessary to obtain a four-dimensional behaviour because the bulk curvature can lead to an *effective compactification*.

2.1. The Randall-Sundrum model

The (second) Randall-Sundrum²⁾ model is based on the following ingredients:

- a five-dimensional bulk spacetime, empty, but endowed with a negative cosmological constant

$$\Lambda = -\frac{6}{\ell^2}, \quad (2.1)$$

- a self-gravitating brane, which represents our world, endowed with a tension σ , and assumed to be Z_2 -symmetric.

The five-dimensional Einstein equations are given by

$$G_{AB} + \Lambda g_{AB} = \kappa^2 T_{AB}, \quad (2.2)$$

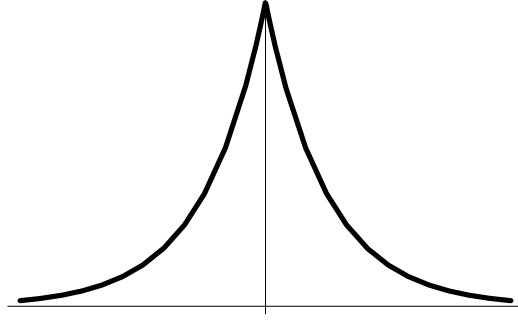
where κ^2 is the gravitational coupling. The corresponding five-dimensional Planck mass, M_5 , can be defined as

$$\kappa^2 = M_5^{-3}. \quad (2.3)$$

The bulk being empty, only the brane contributes to the energy-momentum tensor T_{AB} . There are two equivalent ways of solving Einstein's equations. Either one solves them directly by taking into account the presence of the brane, assumed to be infinitely thin along the extra-dimension, in the form of a *distributional* energy-momentum tensor. Or, one solves first the *vacuum* Einstein equations, i.e. setting the right hand side to zero, and then, one takes into account the brane by imposing appropriate *junction conditions* at the spacetime boundary where the brane is located. These boundary conditions are the generalization, to five dimensions, of the so-called Israel (-Darmois) junction conditions and read

$$[K_{AB}] = -\kappa^2 \left(T_{AB} - \frac{T}{3} h_{AB} \right). \quad (2.4)$$

They relate the jump, between the two sides of the brane, of the extrinsic curvature tensor, defined by $K_{AB} \equiv h_A^C D_C n_B$ (where n^A is the unit vector normal to the brane

Fig. 1. Warping factor $a(y)$.

and $h_{AB} = g_{AB} - n_A n_B$ is the induced metric on the brane), to the brane energy-momentum tensor. For a Z_2 symmetric brane, the jump of the extrinsic curvature is simply twice the value of the extrinsic curvature on one side of the brane.

Provided the tension satisfies the *constraint*

$$\Lambda + \frac{\kappa^4}{6}\sigma^2 = 0, \quad (2.5)$$

which implies in particular that $\sigma = 6M_5^3/\ell$, it can be shown that the five-dimensional Einstein equations admit the following *static* solution

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \quad (2.6)$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric and $a(y)$ is a *warping* scale factor, whose explicit dependence on y is given by

$$a(y) = e^{-|y|/\ell}, \quad (2.7)$$

as shown on Fig. 1. Here, the brane is located at $y = 0$ and the Z_2 symmetry means that y with $-y$ are identified. This bulk solution (2.6-2.7) can also be interpreted as two identical portions of AdS (Anti-de Sitter) spacetime glued together at the brane location.

2.2. Gravity in the Randall-Sundrum model

Let us now investigate the effective gravity in this model, as measured by an observer located on the brane. A first, and rather simple, step is to compute the effective four-dimensional Planck mass. This can be done by substituting in the five-dimensional Einstein-Hilbert action,

$$S_{\text{grav}} = \frac{M_5^3}{2} \int d^4x dy \sqrt{-g} R, \quad (2.8)$$

the metric (2.6) and by integrating over the extra-dimension. One then identifies the factor in front of the resulting four-dimensional Einstein-Hilbert action (for $\eta_{\mu\nu}$) with $M_{Pl}^2/2$, which gives

$$M_{Pl}^2 = M_5^3 \int_{-\infty}^{+\infty} dy a^2(y) = M_5^3 \ell. \quad (2.9)$$

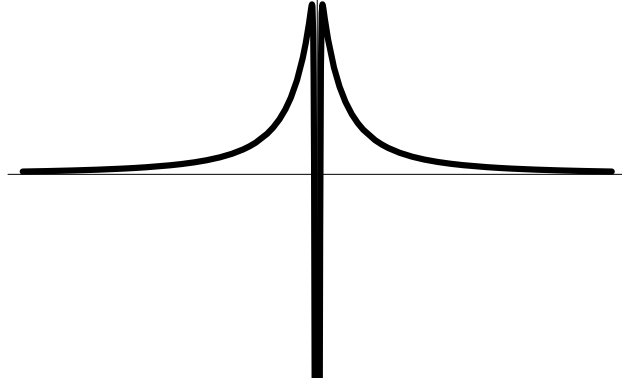


Fig. 2. Effective potential of the Schrödinger-like equation that governs the dependence on the fifth dimension.

It is important to emphasize that the extra-dimension extends here to infinity. In the absence of the warping factor $a(y)$ this would lead to an infinite four-dimensional Planck mass. The warping of the extra-dimension, governed by the AdS lengthscale ℓ , thus leads to an *effective compactification*, even if the extra-dimension is infinite.

To explore further the gravitational behaviour and derive for example the effective potential of a point mass located on the brane, one must study the perturbations about the background metric (2.6). Perturbing the metric, $g_{AB} = \bar{g}_{AB} + h_{AB}$, and working in the gauge $h_{yy} = 0$, $h_{y\mu} = 0$, $h_{\mu}^{\mu} = 0$, $\partial_{\mu}h_{\nu}^{\mu} = 0$, one finds that the linearized Einstein equations reduce to

$$\left(a^{-2} \partial_{(4)}^2 + \partial_y^2 - \frac{4}{\ell^2} + \frac{4}{\ell} \delta(y) \right) h_{\mu\nu} = 0. \quad (2.10)$$

This equation is separable and the solutions can be written as the superposition of eigenmodes $h(x^{\mu}, y) = u_m(y) e^{ip_{\mu}x^{\mu}}$, with $p_{\mu}p^{\mu} = -m^2$. The dependence on the fifth dimension of the modes is governed by a *Schrödinger-like equation*:

$$\frac{d^2 \psi_m}{dz^2} - V(z) \psi_m = -m^2 \psi_m, \quad V(z) = \frac{15}{4(|z| + \ell)^2} - \frac{3}{\ell} \delta(z), \quad (2.11)$$

where $\psi_m = a^{-1/2} u_m$ and $z = \int dy/a(y)$. The potential $V(z)$, plotted in Fig. 2, is “volcano”-shaped and goes to zero at infinity.

One can divide the solutions of this Schrödinger-like equation into:

- a zero mode ($m = 0$), $u_0(y) = a^2(y)/\sqrt{\ell}$, which is concentrated near the brane and reproduces the usual behaviour of 4D gravity;
- a continuum of massive modes ($m > 0$), which are weakly coupled to the brane and modify standard 4D gravity.

More specifically, the perturbed metric outside a spherical source of mass M , and for $r \gg \ell$, is given by³⁾

$$\bar{h}_{00} \simeq \frac{2GM}{r} \left(1 + \frac{2\ell^2}{3r^2} \right), \quad \bar{h}_{ij} \simeq \frac{2GM}{r} \left(1 + \frac{\ell^2}{3r^2} \right), \quad (2.12)$$

where the bar here means that the perturbations are expressed in the Gaussian Normal gauge (i.e. $h_{yy} = h_{y\mu} = 0$ and the brane is located at $y = 0$) and thus correspond directly to the quantities measured on the brane. Standard gravity is thus recovered on scales $r \gg \ell$!

On scales of the order of ℓ , and below, one expects deviations from the usual Newton's law. Since gravity experiments⁴⁾ have confirmed the standard Newton's law down to scales of the order 0.1 mm, this implies

$$\ell \lesssim 0.1\text{mm}, \quad (2.13)$$

and thus $M_{(5)} \gtrsim 10^8$ GeV.

Although the above results apply to linearized gravity, other works, based on second order calculations or numerical gravity, have confirmed the recovery of standard gravity on scales larger than ℓ . However, the behaviour of black holes in the Randall-Sundrum model might significantly deviate from the standard picture. Indeed, inspired by the AdS/CFT correspondence, it has been conjectured that Randall-Sundrum black holes should *evaporate classically*, or, in other words, be classically unstable. The underlying argument is that the five-dimensional classical solutions should correspond to *quantum-corrected* four-dimensional black hole solutions, of a conformal field theory (CFT) coupled to gravity.^{5),6)} Since there are many CFT degrees of freedom into which the black hole can radiate, its life time is shorter than that of a standard black hole:

$$\tau \simeq 10^2 (M/M_\odot)^3 (\ell/1\text{mm})^{-2} \text{years}. \quad (2.14)$$

§3. Homogeneous brane cosmology

Let us now discuss the cosmology of a brane embedded in a five-dimensional bulk spacetime.

3.1. The model

As in standard cosmology, *homogeneity* and *isotropy* along the three ordinary spatial dimensions are assumed. The bulk spacetime is thus required to satisfy the *cosmological symmetry*, i.e. one can foliate the bulk with maximally symmetric three-dimensional surfaces. Note that this is in complete analogy with *the spherical symmetry*, associated with maximally symmetric two-dimensional surfaces in a 4D spacetime.

In addition to the three ordinary spatial dimensions, spanning the homogeneous and isotropic surfaces, one introduces a time coordinate t and a spatial coordinate y for the extra dimension. The cosmological symmetry implies that the metric components depend only on t and y . It is convenient to work in a Gaussian Normal (GN) coordinate system, in which the brane is always located at $y = 0$ and the five-dimensional metric takes the form

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)d\Sigma_k^2 + dy^2, \quad (3.1)$$

where $d\Sigma_k^2$ is the metric for the maximally symmetric three-surface ($k = 0, \pm 1$). Note that, in closer analogy with the spherical symmetry mentioned above, another

possibility would be to choose a coordinate system in which the metric reads

$$ds^2 = -n^2(t, r)dt^2 + b^2(t, r)dr^2 + r^2 d\Sigma_k^2. \quad (3.2)$$

To obtain the equations governing the cosmological evolution, one substitutes the ansatz (3.1) into the five-dimensional Einstein equations

$$G_{AB} + \Lambda g_{AB} = \kappa^2 T_{AB} \quad (3.3)$$

where the energy-momentum tensor, assuming the bulk empty, is only due to the brane matter and thus given by

$$T_A^B = \text{Diag}(-\rho_b(t), P_b(t), P_b(t), P_b(t), 0) \delta(y), \quad (3.4)$$

where ρ_b and P_b are respectively the total energy density and pressure in the brane. The five-dimensional Einstein equations can be solved explicitly⁷⁾ and one gets a solution for the metric components $n(t, y)$ and $a(t, y)$, in terms of $\rho_b(t)$ and $P_b(t)$, defined up to an integration constant.

3.2. The cosmological evolution in the brane

On the brane, the metric is given by

$$ds_b^2 = -n_b(t)^2 dt^2 + a_b(t)^2 d\Sigma_k^2, \quad n_b(t) \equiv n(t, 0), \quad a_b(t) \equiv a(t, 0). \quad (3.5)$$

It can be shown that the scale factor $a_b(t)$ satisfies the *modified Friedmann equation*:⁷⁾

$$H_b^2 \equiv \frac{\dot{a}_b^2}{a_b^2} = \frac{\Lambda}{6} + \frac{\kappa^4}{36} \rho_b^2 + \frac{\mathcal{C}}{a_b^4} - \frac{k}{a_b^2}, \quad (3.6)$$

where \mathcal{C} is an integration constant. It can also be shown that, for an empty bulk, the usual conservation equation holds, which implies

$$\dot{\rho}_b + 3H_b(\rho_b + P_b) = 0. \quad (3.7)$$

For $\Lambda = 0$ and $\mathcal{C} = 0$, the bulk is 5-D Minkowski and the cosmology is highly unconventional since the Hubble parameter is proportional to the brane energy density.⁸⁾ This is incompatible with the standard nucleosynthesis scenario, which depends sensitively on the expansion rate.

To obtain a viable brane cosmology scenario, the simplest way is to generalize the Randall-Sundrum model to cosmology.⁹⁾ In the static version of the previous section, the energy density of the ‘‘Minkowski’’ brane was $\rho_b = \sigma_{RS} \equiv 6M_5^3/\ell$. This can be generalized to an FLRW brane by adding to the intrinsic tension σ_{RS} the usual cosmological energy density $\rho(t)$ so that the total energy density is given by

$$\rho_b(t) = \sigma_{RS} + \rho(t). \quad (3.8)$$

Moreover, the bulk is assumed to be endowed with a negative cosmological constant $\Lambda < 0$, satisfying the constraint (2.5).

Substituting the decomposition (3.8) into the Friedmann equation (3.6), one finds

$$H_b^2 = \frac{8\pi G}{3}\rho + \frac{\kappa^4}{36}\rho^2 + \frac{\mathcal{C}}{a_b^4} - \frac{k}{a_b^2}. \quad (3.9)$$

In the expansion in ρ , the constant term vanishes because of the constraint (2.5), whereas the coefficient of the linear term is the standard one because $8\pi G \equiv \kappa^4\sigma/6$, as implied by (2.5) and (2.9). However, the Friedmann equation (3.9) is characterized by two new features:

- a ρ^2 term, which dominates at high energy;
- a radiation-like term, \mathcal{C}/a_b^4 , usually called *dark radiation*.

The cosmological evolution undergoes a transition from a high energy regime, $\rho \gg \sigma$, characterized by an unconventional behaviour of the scale factor, into a low energy regime which reproduces our standard cosmology. For $\mathcal{C} = 0$, $k = 0$ and an equation of state $w = P/\rho = \text{const}$, one can solve analytically the evolution equations and one finds

$$a(t) \propto t^{1/q} \left(1 + \frac{qt}{2\ell}\right)^{1/q}, \quad q = 3(1+w). \quad (3.10)$$

One clearly sees the transition, at the epoch $t \sim \ell$, between the early, unconventional, evolution $a \sim t^{1/q}$ and the standard evolution $a \sim t^{2/q}$.

In order to be compatible with the nucleosynthesis scenario, the high energy regime, where the cosmological evolution is unconventional, must take place before nucleosynthesis. This requires $\sigma^{1/4} \gtrsim 1$ MeV, and since $\sigma = 6/(\kappa^2\ell) = 6M_5^6/M_P^2$, this gives the constraint $M_5 \gtrsim 10^4$ GeV. This is much less stringent than the constraint from small-scale gravity experiments, which presently require $\ell \lesssim 0.1$ mm and $M_5 \gtrsim 10^8$ GeV. As will be detailed in the next section, another observational constraint applies to the dark radiation constant \mathcal{C} .

3.3. Another point of view

If, instead of the GN ansatz (3.1) for the metric, one starts from the metric (3.2), in analogy with the spherical symmetry, one can use the generalization of the Birkhoff theorem, which states that a vacuum spherical symmetric solution of Einstein's equation is necessarily static and its geometry is Schwarzschild: the 5D vacuum cosmologically symmetric solution of 5D Einstein's equations with a (negative) cosmological constant is necessarily static and corresponds to the AdS-Schwarzschild metric in five dimensions:

$$ds^2 = -f(R)dT^2 + \frac{dR^2}{f(R)} + R^2 d\Sigma_k^2, \quad f(R) = k + \frac{R^2}{\ell^2} - \frac{\mathcal{C}}{R^2}, \quad k = 0, \pm 1. \quad (3.11)$$

In this coordinate system, the brane is *moving* and the modified Friedmann equation obtained above can be recovered from the junction conditions (2.4).^{10), 11)}

§4. Dark radiation

So far, the bulk has been assumed to be *strictly empty*, apart from the presence of the brane. However, the fluctuations of brane matter generate bulk gravitational

waves. Equivalently, the scattering of brane particles produce bulk gravitons ($\psi + \bar{\psi} \rightarrow G$). Therefore, a realistic model must take into account the presence of these bulk gravitons that are emitted by the brane and then propagate in the bulk.

The rate of emission of these gravitons by the brane can be computed explicitly when the brane matter is in thermal equilibrium (with a temperature T). The corresponding energy loss rate is given by^{(12), (13)}

$$\dot{\rho} + 4H\rho = -\frac{315}{512\pi^3} \hat{g} \kappa^2 T^8, \quad (4.1)$$

with an effective number of degrees of freedom given by the following weighted sum of scalar, vector and fermionic degrees of freedom:

$$\hat{g} = (2/3)g_s + 4g_v + g_f. \quad (4.2)$$

The energy transfer from the brane into the bulk modifies the cosmological evolution of the brane, on the one hand because the evolution of the energy density of the brane is modified, on the other hand because the bulk geometry is affected by the gravitons. One can treat self-consistently such an energy transfer by using a five-dimensional generalization of the Vaidya solution.

4.1. Vaidya model

Let us start with the following metric, which generalizes the Vaidya metric to a five-dimensional bulk with a (negative) cosmological constant:

$$ds^2 = -\left(k + \frac{r^2}{\ell^2} - \frac{\mathcal{C}(v)}{r^2}\right) dv^2 + 2drdv + r^2 d\mathbf{x}^2, \quad (4.3)$$

where v is a null coordinate and $\mathcal{C}(v)$ is a function that generalizes the constant \mathcal{C} of the AdS-Schwarzschild metric. If \mathcal{C} is constant, one recovers (3.11) via a change of coordinate.

In the general case, the above metric (4.3) is a solution of the five-dimensional Einstein equations, with a null bulk energy-momentum tensor, i.e. of the form

$$T_{ab} = \psi k_a k_b, \quad k_c k^c = 0. \quad (4.4)$$

One can then show that the cosmological brane evolution is completely determined by the following coupled system⁽¹²⁾

$$\begin{aligned} \frac{d\hat{\rho}}{d\hat{t}} + 4\hat{H}\hat{\rho} &= -\alpha\hat{\rho}^2, \\ \hat{H}^2 &= 2\hat{\rho} + \hat{\rho}^2 + \frac{\hat{\mathcal{C}}}{a^4}, \\ \frac{d\hat{\mathcal{C}}}{d\hat{t}} &= 2\alpha a^4 \hat{\rho}^2 \left(1 + \hat{\rho} - \hat{H}\right), \end{aligned}$$

for the dimensionless quantities $\hat{\rho} = \rho/\sigma_{RS}$, $\hat{t} = \ell t$, $\hat{H} = H\ell$ and $\hat{\mathcal{C}} = \mathcal{C}\ell^2$. The first two equations, the energy non-conservation and the Friedmann equation, are a consequence of the junction conditions whereas the third equation follows from the

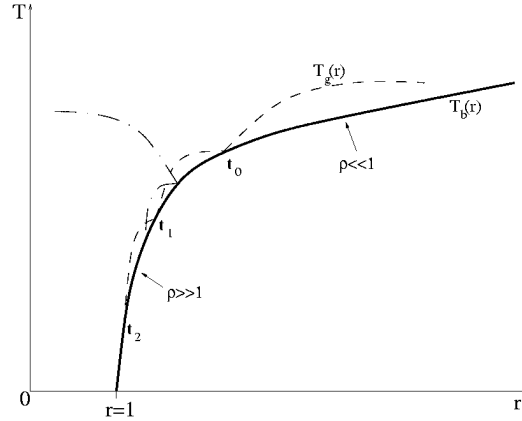


Fig. 3. Examples of graviton trajectories in the bulk (dashed line). The brane trajectory, with relativistic matter, is also shown (continuous line).

Einstein equations. This system can be solved analytically¹⁴⁾ and one finds that, in the low energy regime, \mathcal{C} tends toward a constant, which means that the production of bulk gravitons can be neglected. Although this model is rather nice (and has been generalized to non- Z_2 symmetric branes¹⁵⁾), it is not realistic because it implicitly assumes that the bulk gravitons must be emitted radially, which is not the case.

4.2. More realistic treatment

After their emission, the gravitons propagate freely in the bulk where they follow geodesic trajectories. As illustrated in Fig. 3, some of these gravitons (in fact many) tend to come back onto the brane and bounce off it. All these gravitons contribute to an effective bulk energy-momentum tensor, which can be written as

$$T_{AB}^{(\text{bulk})} = \int d^5p \, \delta(p_M p^M) \sqrt{-g} f p_A p_B, \quad (4.5)$$

where f is the phase space distribution function.

From the 5D Einstein equations, one can derive effective 4D Einstein equations,¹⁶⁾ which in the homogeneous case yield

- the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \left[\left(1 + \frac{\rho}{2\sigma}\right) \rho + \rho_D \right], \quad (4.6)$$

- the non-conservation equation for brane matter, which must be identified with (4.1),

$$\dot{\rho} + 3H(\rho + p) = 2T_{RS}^{(\text{bulk})} n^R u^S, \quad (4.7)$$

where n^A is the unit vector normal to the brane and u^A its velocity in the bulk;

- the non-conservation equation for the “dark” component ρ_D (which includes all effective contributions from the bulk):

$$\dot{\rho}_D + 4H\rho_D = -2 \left(1 + \frac{\rho}{\sigma}\right) T_{AB}^{(\text{bulk})} u^A n^B - 2H\ell T_{AB}^{(\text{bulk})} n^A n^B. \quad (4.8)$$

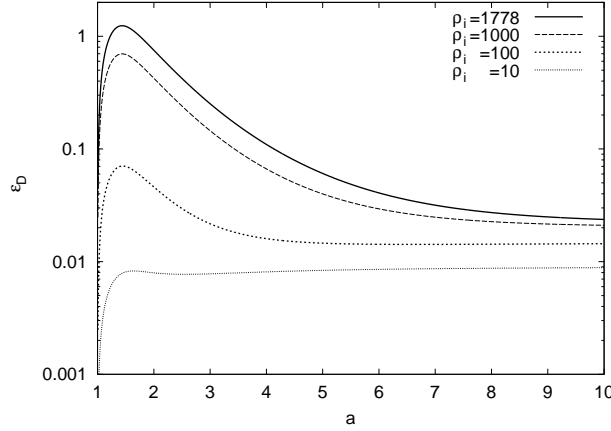


Fig. 4. Evolution of the ratio $\epsilon_D = \rho_D/\rho$ for different values of the initial energy density on the brane ρ_i (in units of σ^4).¹⁷⁾

On the right hand side of this last equation, we find two terms involving the bulk energy-momentum tensor: the first term, due to the energy flux from the brane into the bulk, contributes positively and thus increases the amount of dark radiation whereas the second term, due to the pressure along the fifth dimension, decreases the amount of dark radiation. These terms can be estimated numerically.¹⁷⁾ A striking property is that the gravitons coming back onto the brane and bouncing off it give a significant contribution to the transverse pressure effect, which almost, although not quite, compensates the flux effect. The evolution of the dark radiation, or rather its ratio with respect to the brane radiation density $\epsilon_D \equiv \rho_D/\rho$, is plotted on Fig. 4. At late times, i.e. far in the low energy regime, the ratio reaches a plateau, because the right hand side of (4.8) becomes negligible. The dark component then scales exactly like radiation.

4.3. Observational constraints

The computed amount of dark radiation can be confronted to observations. Indeed, since dark radiation behaves as radiation, it must satisfy the nucleosynthesis constraint on the number of *additional relativistic degrees of freedom*, usually expressed in terms of the extra number of light neutrinos ΔN_ν . The relation between ΔN_ν and ϵ_D is given by

$$\epsilon_D = \frac{7}{43} \left(\frac{g_*}{g_*^{\text{nucl}}} \right)^{1/3} \Delta N_\nu, \quad (4.9)$$

where $g_*^{\text{nucl}} = 10.75$ is the number of degrees of freedom at nucleosynthesis (in fact before the electron-positron annihilation). Assuming $g_* = 106.75$ (standard model), this gives $\epsilon_D \simeq 0.35 \Delta N_\nu$. The typical constraint from nucleosynthesis $\Delta N_\nu \lesssim 0.2$ thus implies

$$\epsilon_D \equiv \frac{\rho_D}{\rho_r} \lesssim 0.03 \left(\frac{g_*}{g_*^{\text{nucl}}} \right)^{1/3}, \quad (4.10)$$

which gives $\epsilon_D \lesssim 0.09$ with the degrees of freedom of the standard model.

§5. Anisotropic brane cosmology

The homogeneous and isotropic brane cosmology is in fact very simple because of the generalized Birkhoff's theorem mentioned earlier. But, when the cosmological symmetry are relaxed, things become rather difficult because the bulk geometry is no longer Schwarzschild-AdS. As a first step towards the general case, it is instructive to study configurations where the cosmology in the brane is homogeneous but anisotropic, e.g. of the Bianchi I type with a metric of the form

$$ds_b^2 = -d\tau^2 + \sum_{i=1}^3 a_i^2(\tau)(dx^i)^2. \quad (5.1)$$

Although many works in the literature have been devoted to this subject, most of them use the effective four-dimensional equations projected on the brane. It is a more challenging task to solve the 5D Einstein equations for the bulk as well, starting e.g. from an ansatz of the form

$$ds_{\text{bulk}}^2 = -n^2(t, y)dt^2 + \sum_{i=1}^3 a_i^2(t, y)(dx^i)^2 + dy^2. \quad (5.2)$$

Assuming that the metric is separable, it turns out that explicit solutions have been found. They are given by¹⁸⁾

$$ds^2 = \sinh^{1/2}(4y/\ell) \left[-\tanh(2y/\ell)^{2q_0} dt^2 + \sum_i \tanh(2y/\ell)^{2q_i} t^{2p_i} (dx^i)^2 \right] + dy^2, \quad (5.3)$$

where the seven coefficients q_μ and p_i must satisfy the constraints

$$\sum_{\mu=0}^3 q_\mu = 0, \quad \sum_{\mu} q_\mu^2 = \frac{3}{4}, \quad \sum_{i=1}^3 p_i = 1, \quad \sum_i p_i^2 = 1, \quad \sum_i q_i(p_i + 1) = 0. \quad (5.4)$$

In general, a brane embedded in an anisotropic bulk spacetime must contain matter with *anisotropic stress*, because of the junction conditions, which can here be separated into two parts:

- isotropic part:

$$n^{-1} \dot{y}_b \dot{A} \Big|_b + \sqrt{1 + \dot{y}_b^2} A' \Big|_b = \frac{\kappa^2}{6} \rho_b, \quad (5.5)$$

- anisotropic part:

$$n^{-1} \dot{y}_b \dot{B}_i \Big|_b + \sqrt{1 + \dot{y}_b^2} B'_i \Big|_b = \frac{\kappa^2}{2} \pi_i, \quad (5.6)$$

where π_i is the anisotropic pressure in the brane, and using the notation $3A \equiv \ln(a_1 a_2 a_3)$ and $B_i \equiv \ln a_i - A$. Note that the brane position y_b is not assumed to be fixed here: in this sense the coordinate system is not Gaussian Normal. Interestingly, the above solutions include a particular bulk geometry, for $q_0 = \pm\sqrt{3}/4$, in which one can embed a moving brane with perfect fluid as matter, i.e. $\pi_i = 0$. For this particular case, the effective cosmological equation of state $P_{\text{eff}}/\rho_{\text{eff}}$ is negative but goes to zero at late times.

§6. Brane-worlds with generalized gravity theories

So far, we have used five-dimensional Einstein gravity for the bulk. However, one might envisage more general gravity theories to describe the bulk space-time. One possibility is to take into account higher order curvature terms in the five-dimensional action. It turns out that there is a particular combination of the second order curvature terms, called the Gauss-Bonnet term, which yields well-behaved equations of motion. The five-dimensional action with a Gauss-Bonnet term reads^{*)}

$$\mathcal{S} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g_5} \left[-2\Lambda_5 + \mathcal{R} + \alpha \left(\mathcal{R}^2 - 4\mathcal{R}_{ab}\mathcal{R}^{ab} + \mathcal{R}_{abcd}\mathcal{R}^{abcd} \right) \right]$$

All the steps discussed previously can be revisited in this more general context. One can find a Minkowski brane if the brane tension is adjusted to the value

$$\kappa_5^2 \sigma = 2(3 - \beta)/\ell \quad (6.1)$$

where $\beta \equiv 4\alpha/\ell^2$. The effective four-dimensional gravitational constant is given by

$$\kappa_4^2 = \frac{\kappa_5^2}{\ell(1 + \beta)}. \quad (6.2)$$

In the cosmological context, the modified Friedmann equation is now given by²¹⁾

$$\kappa_5^2(\rho + \sigma) = \frac{2}{\ell} \sqrt{1 + H^2 \ell^2} \left[3 + \beta (2H^2 \ell^2 - 1) \right]. \quad (6.3)$$

Assuming that the Gauss-Bonnet term represents a small correction to the Einstein-Hilbert term, i.e. $\beta \ll 1$, this Friedmann equation exhibits three different regimes:

- a Gauss-Bonnet regime for $H\ell \gg \beta^{-1}$, during which

$$H^2 \approx \left[\frac{\kappa_5^2}{4\beta\ell^2} \rho \right]^{2/3}, \quad (6.4)$$

- a five-dimensional Einstein regime for $1 \ll H\ell \ll \beta^{-1}$, with the behaviour

$$H^2 \approx \frac{\kappa_5^2}{36} \rho^2, \quad (6.5)$$

- finally, the ordinary four-dimensional Einstein regime for $H\ell \gg 1$, characterized by the usual Friedmann law

$$H^2 \approx \frac{\kappa_4^2}{3} \rho. \quad (6.6)$$

^{*)} It is interesting to note that this Einstein-Gauss-Bonnet theory has been introduced in a six-dimensional bulk in order to recover Einstein gravity on a codimension 2 brane.¹⁹⁾ This result has been extended to any even dimension in the more general context of Lovelock gravity theories.²⁰⁾

§7. Brane inflation

In brane cosmology, the famous horizon problem is much less severe than in standard cosmology, because the gravitational horizon, associated with the signal propagation in the bulk, can be much bigger than the standard photon horizon, associated with the signal propagation on the brane.²²⁾ However, it is still alive because the energy density on the brane is limited by the Planck limit $\rho \sim M_5^4$. Thus, one must still invoke inflation, although alternative ideas based on the collision of branes²³⁾ have been actively explored*).

The simplest way to get inflation in the brane^{**)} is to detune the brane tension from its Randall-Sundrum value (2.6) in order to obtain a net effective four-dimensional cosmological constant that is positive. This leads to exponential expansion on the brane. In the GN coordinate system, the metric is separable and can be written as

$$ds^2 = A(y)^2 (-dt^2 + e^{2Ht} d\vec{x}^2) + dy^2, \quad (7.1)$$

with

$$A(y) = \cosh \mu y - \left(1 + \frac{\rho}{\sigma}\right) \sinh \mu |y|. \quad (7.2)$$

As in the Randall-Sundrum case, the linearized Einstein equations for the tensor modes lead to a *separable* wave equation. The shape along the fifth dimension of the corresponding massive modes is governed by the Schrödinger-like equation

$$\frac{d^2 \Psi_m}{dz^2} - V(z) \Psi_m = -m^2 \Psi_m, \quad (7.3)$$

after introducing the new variable $z - z_b = \int_0^y d\tilde{y}/A(\tilde{y})$ (with $z_b = H^{-1} \sinh^{-1}(H\ell)$) and the new function $\tilde{\Psi}_m = A^{-1/2} u_m(y)$. The potential is given by

$$V(z) = \frac{15H^2}{4 \sinh^2(Hz)} + \frac{9}{4} H^2 - \frac{3}{\ell} \left(1 + \frac{\rho}{\sigma}\right) \delta(z - z_b) \quad (7.4)$$

and plotted in Fig. 5. In contrast with the Randall-Sundrum potential, the potential goes asymptotically to the non-zero value $9H^2/4$. This indicates the presence of a gap between the zero mode ($m = 0$) and the continuum of Kaluza-Klein modes ($m > 3H/2$).

In practice, inflation is not strictly de Sitter but the de Sitter case discussed above is a good approximation when $\dot{H} \ll H^2$. To get “realistic” inflation in the brane, two main approaches have been considered: either to assume a five-dimensional scalar

*) Note, however, that the generation of a quasi-scale-invariant fluctuation spectrum, as required by observations, remains problematic because the fifth dimension goes to zero when two Z_2 -symmetric branes collide and the evolution of perturbations is then ill-defined. By contrast, the collision of branes that are not both Z_2 -symmetric is well-behaved and precise conservation laws can be derived.²⁴⁾

**) We do not consider here models, also called brane inflation, where the inflaton is the distance between two branes in relative motion with respect to each other. These models are based on an effective four-dimensional approach. In the present context, where the self-gravity of the brane is essential, a four-dimensional approach does not apply²⁵⁾ except in the low-energy limit.²⁶⁾

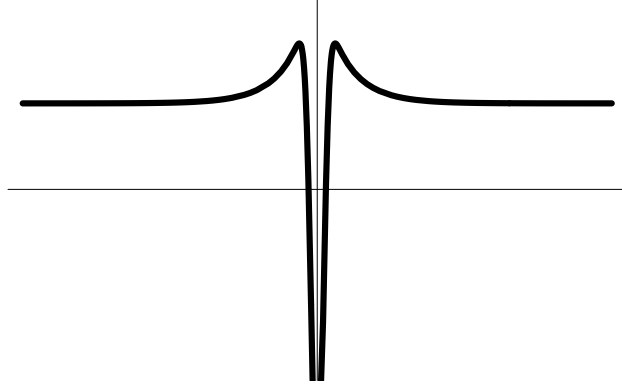


Fig. 5. Potential for the graviton modes in a de Sitter brane

field which induces inflation in the brane,²⁷⁾ or to suppose a four-dimensional scalar field confined on the brane.²⁸⁾

In the latter case, the cosmological evolution during inflation is obtained by substituting the energy density $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ in the modified Friedmann equation (3.9). For slow-roll inflation, this can be approximated by

$$H^2 \simeq \frac{8\pi G}{3} \left(1 + \frac{V}{2\sigma}\right) V. \quad (7.5)$$

Interestingly, because of the modified Friedmann equation, new features appear at high energy ($V > \sigma$): the slow-roll conditions are changed and, because the Hubble parameter is bigger than the standard value, yielding a higher friction on the scalar field, inflation can occur with potentials usually too steep to sustain it.²⁹⁾

The scalar and tensor spectra generated during inflation driven by a brane scalar field have also been computed^{28),30)} (although there might be some subtleties for the scalar modes^{31),32)}). They are modified with respect to the standard results, according to the expressions

$$P_S = P_S^{(4D)} \left(1 + \frac{V}{2\sigma}\right)^3, \quad P_T = P_T^{(4D)} F^2(H\ell), \quad (7.6)$$

with

$$F(x) = \left\{ \sqrt{1+x^2} - x^2 \ln \left[\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right] \right\}^{-1/2}. \quad (7.7)$$

At low energies, i.e. for $H\ell \ll 1$, $F \simeq 1$ and one recovers exactly the usual four-dimensional result but at higher energies the multiplicative factor F provides an *enhancement* of the gravitational wave spectrum amplitude with respect to the four-dimensional result: $F \simeq (3/2)H\ell \sim V/\sigma$ at very high energies, i.e. for $H\ell \gg 1$. Nevertheless, comparing this with the amplitude of the scalar spectrum, one finds that, at high energies ($\rho \gg \sigma$), the *tensor over scalar ratio is in fact suppressed* with respect to the four-dimensional ratio.

These results have been extended to the case of the Einstein-Gauss-Bonnet theory discussed in the previous section.³³⁾ The scalar spectrum is given by

$$\mathcal{P}_S = \mathcal{P}_S^{4D} G_\beta^2(H\ell) \quad (7.8)$$

with

$$G_\beta^2(x) = \left(\frac{3(1+\beta)x^2}{2\sqrt{1+x^2}(3-\beta+2\beta x^2) + 2(\beta-3)} \right)^3 \quad (7.9)$$

whereas the tensor spectrum is given by

$$\mathcal{P}_T = \mathcal{P}_T^{4D} F_\beta^2(H\ell) \quad (7.10)$$

with

$$F_\beta^{-2}(x) = \sqrt{1+x^2} - \left(\frac{1-\beta}{1+\beta} \right) x^2 \sinh^{-1} \frac{1}{x}. \quad (7.11)$$

All these results give the amplitude of the perturbations during inflation. A difficult question is to determine how the perturbations will evolve during the subsequent cosmological phases, the radiation and matter eras.

§8. Cosmological perturbations

A crucial test for brane cosmology is the confrontation with cosmological observations, in particular the CMB fluctuations. Although the *primordial* power spectra for scalar and tensor perturbations have been computed, the subsequent evolution of the cosmological perturbations is non trivial and has not been fully solved yet. Indeed, in contrast with standard cosmology where the evolution of cosmological perturbations can be reduced to ordinary differential equations for the Fourier modes, the evolution equations in brane cosmology are partial differential equations with two variables: the time and the fifth coordinate. Another delicate point is to specify the boundary conditions, both in time and space.

An instructive, although limited, approach for brane cosmological perturbations is the brane point of view, based on the 4D effective Einstein equations on the brane, usually written in the form¹⁶⁾

$$G_{\mu\nu} + \Lambda_4 g_{\mu\nu} = 8\pi G \tau_{\mu\nu} + \kappa^2 \Pi_{\mu\nu} - E_{\mu\nu}, \quad (8.1)$$

where $\tau_{\mu\nu}$ is the brane energy-momentum tensor, $\Pi_{\mu\nu}$ is a tensor depending quadratically on $\tau_{\mu\nu}$ (which gives the ρ^2 term of the Friedmann equation in the homogeneous case), and $E_{\mu\nu}$, which corresponds to the dark radiation in the homogeneous case, is the projection on the brane of the bulk Weyl tensor.

It is then a straightforward exercise, starting from (8.1), to write down explicitly the perturbed effective Einstein equations on the brane, which will look exactly as the four-dimensional ones for the geometrical part but with extra terms due to $\Pi_{\mu\nu}$ and $T_{\mu\nu}^{Weyl}$. One thus gets equations relating the perturbations of the metric to the matter perturbations *and* the perturbations of the projected Weyl tensor, which formally can be assimilated to a virtual fluid, with corresponding (perturbed) energy

density $\rho_E + \delta\rho_E$, pressure $P_E + \delta P_E = \frac{1}{3}(\rho_E + \delta\rho_E)$ and anisotropic pressure.³⁴⁾ The contracted Bianchi identities ($\nabla_\mu G^\mu_\nu = 0$) and energy-momentum conservation for matter on the brane ($\nabla_\mu \tau^\mu_\nu = 0$) ensure, using Eq. (8.1), that

$$\nabla_\mu E^\mu_\nu = \kappa^4 \nabla_\mu \Pi^\mu_\nu. \quad (8.2)$$

In the background, this tells us that ρ_E behaves like radiation, as we knew already, and for the first-order perturbations, one finds that the effective energy of the projected Weyl tensor is conserved independently of the quadratic energy-momentum tensor. The only interaction is a momentum transfer.

It is also possible to construct³⁵⁾ gauge-invariant variables corresponding to the curvature perturbation on hypersurfaces of uniform density, both for the brane matter energy density and for the total effective energy density (including the quadratic terms and the Weyl component). These quantities are extremely useful because their evolution on scales larger than the Hubble radius can be solved easily. However, their connection to the large-angle CMB anisotropies involves the knowledge of anisotropic stresses due to the bulk metric perturbations. This means that *for a quantitative prediction of the CMB anisotropies, even at large scales, one needs to determine the evolution of the bulk perturbations.*

In summary, one can obtain a set of equations for the brane linear perturbations, where one recognizes the ordinary cosmological equations but modified by two types of corrections:

- modification of the homogeneous background coefficients due to the additional ρ^2 terms in the Friedmann equation. These corrections are negligible in the low energy regime $\rho \ll \sigma$;
- presence of source terms in the equations. These terms come from the bulk perturbations and cannot be determined solely from the evolution inside the brane. To determine them, one must solve the full problem in the bulk (which also means to specify some initial conditions in the bulk).

Most of the recent works³⁶⁾ studying the post-inflation evolution of brane cosmological perturbations have concentrated on tensor modes, which are simpler because they are not coupled, like scalar modes, to brane matter fluctuations.

§9. Conclusions

In this contribution, I have presented some aspects of brane-world models, covering both the (static) Randall-Sundrum model and its cosmological extensions. Due to lack of time/space, I have not discussed many other interesting topics in the field. Examples are the brane cosmology of models involving Gauss-Bonnet corrections; the induced gravity models, where one includes a 4D Einstein-Hilbert action for the brane and which can lead to late-time cosmological effects mimicking dark energy.

There are still many open questions in brane cosmology. Even in the simplest set-up, discussed here, based on a cosmological extension of the Randall-Sundrum model, the evolution of cosmological perturbations has not yet been solved, although some significant progress has been made. The situation is still more complicated in more sophisticated models, involving a bulk scalar field and/or collision of branes. It must

be emphasized that the predictions for the cosmological perturbations, as observed in the CMB experiments, and their adequation with the present data, is a crucial test for brane-world models for which the early universe is modified. More direct tests of brane-world models involve gravity experiments or collider experiments. However, if the fundamental Planck mass is too high, such direct experiments cannot see extra-dimensional effects and one must turn to cosmology to try to see indirect signatures from the early universe.

Another direction of research is to make contact between the brane-worlds, which are still only phenomenological models, and a fundamental theory like string theory.

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